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On Regular and Intra-Regular Γ-Semihyperrings-II

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Abstract

In this paper we have proved some results on regular and intra-regular Γ-semihyperring. The characterization of regular and intra-regular Γ-semihyperring with the help of quasi-ideals, bi-ideals, prime bi-ideals and irreducible bi-ideals of Γ-Semihyperrings.

Keywords: Quasi-ideal, Bi-ideal, Regular Γ-Semihyperring, prime bi-ideals, irreducible bi-ideals

1. Introduction

In 1934, the concept of hyper structure theory comes before the word very first when French Mathematician presented paper in conference. In theory of classical algebraic structure, the composition of two elements is an element and in theory of hyperstructure the composition of two element becomes a set. In 2003, Corsini and Leoreanu [1] have given application of theory of hyperstructures in various subjects like: geometry, cryptography, artificial intelligence, relation algebras, automata, median algebras, relation algebras, fuzzy sets and codes. If we let H be a non-empty set. Then, the map $o : H \times H \longrightarrow P^*(H)$ is called a hyperopertion, where $P^*(H)$ is the family of all non-empty subsets of H and the couple (H, o) is called a hypergroupoid. Moreover, the couple (H, o) is called a semihypergroup if for every $a, b, c \in H$ we have, $(aob)o\ c = ao(boc)$.

As theory of hyperstructure has vast application in various fields of sciences so it is essential to study the concepts of classical algebraic structure in hyperstructure theory. The main aim of this paper is to study the concepts of classical algebraic structure to a hyperstructure theory. Jagatap and Pawar [2] introduced intra-regular Γ-Semiring and made its characterizations with the help of ideals Γ-Semirings.

Pawar *et al.* [3] introduced the notion of regular Γ-semihyperring. Patil and Pawar [4, 5] introduced and studied quasi-ideals and bi-ideals of a Γ-semihyperring. Patil [6] has characterised the regular and intra-regular Γ-Semihyperrings. Here we add some more results on regular and intra-regular Γ-Semihyperrings and made its characterizations with the help of prime bi-ideals and irreducible bi-ideals of Γ -Semihyperrings. of Γ-Semihyperrings analogues to [7,8]. Ostadhadi-Dehkordi and Davvaz [7] studied ideal theory in Γ-Semihyperrings. In this paper regular and intra-regular Γ-Semihyperring has been studied on the line of [7,8]. In section 2, Preliminaries are given required to understand the paper. In section 3, we have characterized regular and intra-regular Γ-Semihyperring.

2. Preliminaries

Here are some useful definitions and the readers are requested to refer [7-9].

Definition 2.1. [7] Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \longrightarrow P^*(H)$ (images to be denoted by $a\alpha b$, for all $a, b \in R$ and $\in \Gamma$) and $P^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$.

(2) $(a + b)\alpha c = a\alpha c + b\alpha c$.

(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$.

(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.2. [7] A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$ for all $a, b \in R$ and $\alpha \in \Gamma$.

Definition 2.3. [7] A Γ-semihyperring R is said to be with zero, if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a$, $0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$, then

$$A + B = \{x | x \in a + b, a \in A, b \in B\}$$

$$A\Gamma B = \{x \,| x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}$$

Definition 2.4. [7] A non-empty subset $R_1$of Γ-semihyperring R is called a Γ -subsemihyperring if it is closed with respect to the multiplication and addition, that is, $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Definition 2.5. [7] A right (left) ideal I of a Γ-semihyperring R is an additive sub semihypergroup of $(R, +)$ such that $I\Gamma R \subseteq I$ ($R\Gamma I \subseteq I$). If I is both right and left ideal of R, then we say that I is a two sided ideal or simply an ideal of R.

Definition 2.6. [5] A non-empty set B of Γ-semihyperring R is a bi-ideal of R if B is a Γ-subsemihyperring of R and $B\Gamma R\Gamma B \subseteq B$.

Definition 2.7. [4] A subsemihypergroup Q of $(R, +)$ is said to be a quasi-ideal of Γ -semihyperring R if $(R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$.

Definition 2.8. [3] An element e of Γ-semihyperring R is said to be a left (right) identity of R if $r \in e\alpha r$ ($r \in r\alpha e$) for all, $r \in R$ and $\alpha \in \Gamma$.

An element e of Γ-semihyperring R is said to be a two sided identity or simply an identity if e is both left and right identity, that is $r \in e\alpha r \cap r\alpha e$) for all $r \in R$ and $\alpha \in \Gamma$

Theorem 2.9. [4] Every quasi-ideal of a Γ-Semihyperring R is a bi-ideal of R.

Theorem 2.10. [5] Every one sided (two sided) ideal of a Γ-Semihyperring R is a bi-ideal of R.

Theorem 2.11. [4] Every one sided (two sided) ideal of a Γ-Semihyperring R is a quasi-ideal of R.

Definition 2.12. [10] A Γ-Semihyperring S is said to be an intra-regular Γ-Semihyperring if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.

Theorem 2.13. [10] Let S be a Γ-Semihyperring. Then S is intra-regular if and only if each right ideal R left ideal L of S satisfies $R \cap L \subseteq L\Gamma R$.

Definition 2.14. [6] A Γ-Semihyperring S is said to be a regular Γ-Semihyperring if for any $x \in S$, $x \in x\Gamma S\Gamma x$.

Theorem 2.15. [6] Let S be a Γ-Semihyperring with an identity. Then S is regular if and only if each right ideal R left ideal L of S satisfies $R \cap L = R\Gamma L$.

Theorem 2.16. [4] Intersection of left ideal and right ideal of a Γ-Semihyperring S is a quasi-ideal of S.

3. Regular and Intra-regular Γ-Semihyperring

In this section, we studied regular and intra-regular Γ-Semihyperring and made its characterization with the help of different ideals of Γ- Semihyperring on the line of Jagatap and Pawar [2]. Throughout this paper we consider that Γ-Semihyperring has identity element.

Theorem 3.1. Following statements are equivalent in Γ-Semihyperring S.

1. S is regular and intra-regular Γ-Semihyperring.
2. For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.
3. For quasi-ideal Q and a bi-ideal B of S, $Q \cap B \subseteq (B\Gamma Q\Gamma B) \cap (Q\Gamma B\Gamma Q)$.
4. For quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2\Gamma Q_1) \cap (Q_2\Gamma Q_1\Gamma Q_2)$.

Proof. (1) ⇒ (2)

Suppose S is a regular and intra-regular. Let $x \in B_1 \cap B_2$, for any bi-ideals B_1 and B_2 of S. As S is regular and intra-regular, we have $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$ respectively. Then

$x\Gamma S\Gamma x \subseteq x\Gamma S\Gamma(x\Gamma S\Gamma x)$ (Since S a regular)

$\subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma(x\Gamma S\Gamma x)$

$\subseteq (x\Gamma S\Gamma(S\Gamma x\Gamma x\Gamma S))\Gamma S\Gamma((S\Gamma x\Gamma x\Gamma S)\Gamma S\Gamma x)$ (Since S an intra-regular)

$\subseteq (x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x)$ (Since $x \in Q \cap R \cap L$)

$\subseteq (B_1\Gamma S\Gamma B_1)\Gamma(B_2\Gamma S\Gamma B_2)\Gamma(B_1\Gamma S\Gamma B_1)$

$\subseteq (B_1\Gamma B_2\Gamma B_1)$

Thus we get, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1)$. Similarly we can show that, $B_1 \cap B_2 \subseteq (B_2\Gamma B_1\Gamma B_2)$. Hence we get,

$B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.

(2) ⇒ (3) and (3) ⇒ (4) As every quasi-ideal is bi-ideal by Theorem 2.9. Implication follows easily.

(4) ⇒ (1) Let R be a right-ideal and L be a left ideal of a Γ-Semihyperring S. By Theorem 2.16. $R \cap L$ Is a quasi-ideal of S. By (4), we have $(R \cap L) = (R \cap L)\Gamma(R \cap L)\Gamma(R \cap L) \subseteq R\Gamma L\Gamma L \subseteq R\Gamma L$. But $R\Gamma L \subseteq R \cap L$ always holds. Thus we get, $R \cap L = R\Gamma L$. Thus we get S is a regular and intra regular Γ-Semihyperring by Theorem 2.13 and 2.15.

Theorem 3.2. Following statements are equivalent in Γ-Semihyperring S.

1. S is regular and intra-regular Γ-Semihyperring.
2. For bi-ideals B and left ideal L of S, $B \cap L \subseteq (B\Gamma L\Gamma B) \cap (L\Gamma B\Gamma L)$.
3. For quasi-ideal Q and left ideal L of S, $Q \cap L \subseteq (Q\Gamma L\Gamma Q) \cap (L\Gamma Q\Gamma L)$.

Proof. Every one side ideals are bi-ideal so proof is obvious by Theorem 3.1.

Theorem 3.3. Following statements are equivalent in Γ-Semihyperring S.

1. S is regular and intra-regular Γ-Semihyperring.
2. For bi-ideals B and right ideal R of S, $B \cap R \subseteq (B\Gamma R\Gamma B) \cap (R\Gamma B\Gamma R)$.
3. For quasi-ideal Q and right ideal R of S, $Q \cap R \subseteq (Q\Gamma R\Gamma Q) \cap (R\Gamma Q\Gamma R)$.

Proof. Every one side ideals are bi-ideal so proof is obvious by Theorem 3.1.

One may define generalized bi-ideal of Γ-Semihyperring S as a non-empty subset B of a Γ-Semihyperring S satisfying $B\Gamma S\Gamma B \subseteq B$.

Theorem 3.4. S is regular and intra-regular Γ-Semihyperring if and only if for quasi-ideal Q and a generalized bi-ideal G of S, $G \cap Q \subseteq (G\Gamma Q)\cap(Q\Gamma G)$.

Proof. Suppose S is regular and intra-regular. Let G be generalized bi-ideal and Q be a quasi-ideal of S. Take any $x \in G \cap Q$. Then we have, $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$ respectively. Therefore

$x\Gamma S\Gamma x \subseteq x\Gamma S\Gamma(x\Gamma S\Gamma x)$ (Since S is a regular)

$\subseteq x\Gamma S\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma S\Gamma x$

$= (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\,\Gamma S\Gamma x)$

$\subseteq (x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x)$

$\subseteq (G\Gamma S\Gamma G)\Gamma(Q\Gamma S\Gamma Q)$

As S is a regular by Theorem 3.1, we have $Q\Gamma S\Gamma Q = Q$. Also, $G\Gamma S\Gamma G \subseteq G$, since G is a generalised bi-ideal. Therefore we get, $x\Gamma S\Gamma x \subseteq G\Gamma$Q. Similarly we can show, , $x\Gamma S\Gamma x \subseteq Q\Gamma$G. Thus we get, $G \cap Q \subseteq (G\Gamma Q)\cap(Q\Gamma G)$.

Conversely, assume that for quasi-ideal Q and a generalized bi-ideal G of S, $G \cap Q \subseteq (G\Gamma Q)\cap(Q\Gamma G)$. Let L be a left ideal and R be a right ideal of Γ-Semihyperring S. Then R is generalised bi-ideal and L is a quasi-ideal of S. Then $R \cap L \subseteq (R\Gamma L)\cap(L\Gamma R) \subseteq L\Gamma R$. Similarly $R \cap L \subseteq (R\Gamma L)\cap(L\Gamma R) \subseteq R\Gamma L$. Thus we get by Theorem 2.13. S is intra-regular Γ-Semihyperring. Also we have $R\Gamma L \subseteq R \cap L$ is always holds. Therefore $R \cap L = R\Gamma$L. Hence by Theorem 2.15. S is a regular Γ-Semihyperring.

Definition 3.5. A proper bi-ideal B of Γ-Semihyperring S is called an irreducible bi-ideal if $B_1 \cap B_2 = B$ implies $B_1 = B$ or $B_2 = B$ for any bi-ideals B_1 and B_2 of S.

Definition 3.6 A proper bi-ideal B of Γ-Semihyperring S is called a strongly irreducible bi-ideal if $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S.

Definition 3.7. A proper bi-ideal B of S is called a prime bi-ideal if $B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S.

Definition 3.8. A proper bi-ideal B of S is called a strongly prime bi-ideal if $(B_1\Gamma B_2)\cap(B_2\Gamma B_1) \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S.

Definition 3.9. A proper bi-ideal B of S is called a semiprime bi-ideal if $B_1\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S.

Theorem 3.10. If B is a bi-ideal of S and $a \in S$ such that $a \notin B$, then there exists an irreducible bi-ideal I of S such that $B \subseteq I$ and $a \notin I$.

Theorem 3.11. Any proper bi-ideals of B of S is the intersection of all irreducible bi-ideals of S containing B.

Theorem 3.12. Following statements are equivalent in Γ-Semihyperring S.

1. S is regular and intra-regular Γ-Semihyperring.
2. For bi-ideal B of S, $B\Gamma$B = B.
3. $B_1 \cap B_2 = (B_1\Gamma B_2)\cap(B_2\Gamma B_1)$ for any bi-deals B_1 and B_2 of S.
4. Each bi-ideal of S is semiprime
5. Each proper bi-ideal of S is the intersection of irreducible semiprime bi-ideals of S which contain it.

Proof. (1) ⇔ (2) is obvious.

(2) ⟹ (3) Suppose that For any bi-ideal B of S, $B\Gamma$B = B. Let B_1 and B_2 be any two bi-ideals of S. As $B_1 \cap B_2$ is a bi-ideal of S. By (2), $B_1 \cap B_2 = (B_1 \cap B_2)\Gamma(B_1 \cap B_2) \subseteq B_1\Gamma B_2$. Similarly $B_1 \cap B_2 = (B_1 \cap B_2)\Gamma(B_1 \cap B_2) \subseteq B_2\Gamma B_1$. Thus we get, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2)\cap(B_2\Gamma B_1)$. Also, $B_1\Gamma B_2$ and $B_2\Gamma B_1$ are bi-ideals of S. We can show,$(B_1\Gamma B_2)\cap(B_2\Gamma B_1) \subseteq B_1$, $(B_1\Gamma B_2)\cap(B_2\Gamma B_1) \subseteq B_2$. Thus we get, $(B_1\Gamma B_2)\cap(B_2\Gamma B_1) \subseteq B_1 \cap B_2$. So from both inclusion we get, $B_1 \cap B_2 = (B_1\Gamma B_2)\cap(B_2\Gamma B_1)$ for any bi-deals B_1 and B_2 of S.
(3) ⟹ (4)

Let B be any bi-ideal of S. Assume $B_1\Gamma B_1 \subseteq B$ for any bi-ideal B_1 os S. Therefore by (3), we have $B_1 = B_1 \cap B_1 = (B_1\Gamma B_1)\cap(B_1\Gamma B_1) = B_1\Gamma B_1 \subseteq B$. Thus we get each bi-ideal is semiprime

(4) $\Longrightarrow$ (5)

By Theorem 3.11 and by (4) implication easily follows.

(5) $\Longrightarrow$ (2) Let B be a bi-ideal of S. If $B\Gamma B = S$ then result is obvious. Suppose $B\Gamma B \neq S$. Then $B\Gamma B$ is a proper bi-ideal of S. Hence by (5) it is the intersection of irreducible semiprime bi-ideals of S which contain it. $B\Gamma B = \cap_{i\in\Delta} \{B_i | B_i \text{is irreducible semiprime bi} - \text{ideal}\}$, where Δ is an indexing set. Therefore $B\Gamma B \subseteq B_i$ for all $i \in \Delta$. As each B_i is a semiprime ideal we have $B \subseteq B_i$ for all $i \in \Delta$. Thus we get $B \subseteq B\Gamma B$. But $B\Gamma B \subseteq B$ always holds. So we get, $B\Gamma B = B$.

Thus all implications holds.

Theorem 3.13. If S is regular and intra-regular Γ-Semihyperring, then a bi-ideal B of S is a strongly irreducible bi-ideal if and only if B is a strongly prime bi-ideal.

4. Conclusion

In this paper we studied regular and intra-regular Γ-Semihyperring as extension of the paper On Regular and Intra-Regular Γ-Semihyperrings. The characterization of regular and intra-regular Γ-semihyperring with the help of quasi-ideals, bi-ideals, prime bi-ideals and irreducible bi ideals of Γ-Semihyperrings. It is found that there is lot of scope to study prime (strongly Prime) bi-ideal irreducible (strongly irreducible) bi-ideal of Γ-semihyperring.

Conflicts of interest: The author stated that no conflicts of interest.

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